CH2 PATH INTEGRALS

2.1 Hamiltonian formulation

→ In QFT2, we computed the amplitudes $\langle q' | \hat{U} | q \rangle$ with the evolution operator $\hat{U} = \exp\left\{ \frac{-i}{\hbar} \hat{H} . (t - t') \right\}$

Not | We write $\langle q'; t' | q; t \rangle \equiv \langle q' | \hat{U}(t', t) | q \rangle$

PROP | The path integrals formulation is the following:

$$\langle q'; t' | q; t \rangle = \int_{q(t)=q}^{q(t')=q'} \prod_{a,\tau} dq^a(\tau) \prod_{b,\varsigma} \frac{dp^b(\varsigma)}{2\pi\hbar} \exp\left\{ \frac{i}{\hbar} S_H[q,p] \right\}$$

with $S_H[q,p] = \int_t^{t'} d\tau \left(\dot{q}^c p_c - \mathcal{H}[q,p] \right)$ is the 1st order Hamiltonian action

2.2 Partition function

→ The partition function is given by:

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta \hat{H}} = \int dq \langle q | e^{-\beta \hat{H}} | q \rangle$$

$$= \int \prod_{a,\tau} dq^a(\tau) \prod_{b,\varsigma} \frac{dp^b(\varsigma)}{2\pi\hbar} \exp\left\{ \frac{-1}{\hbar} S_H^E[q,p] \right\}$$

with $S_H^E \equiv \int_0^{\hbar\beta} d\tau_E \left\{ -i \dot{q}^a(\tau_E) p_a(\tau_E) + H[q(\tau_E), p(\tau_E)] \right\}$

↳ Indeed: $\mathrm{Tr}\, \hat{U}(t', t) = \int dq \langle q | e^{-i/\hbar \hat{H}(t',t)} | q \rangle$

$$= \int_{\text{periodic path}} \mathcal{D}q \, \mathcal{D}p \, \frac{1}{2\pi\hbar} e^{i/\hbar S_H[q,p]}$$

where $q(0) = q(T)$; $p(0) = p(T)$, $T \equiv t' - t$, and $S_H = \int_t^{t'} d\tau \left(p_a \dot{q}^a - H(p,q) \right)$

→ We set $t' - t = -i\hbar\beta$ (Wick rotation) $\Rightarrow e^{-i/\hbar H(t'-t)} \mapsto e^{-\hat{H}\beta}$

→ To get $i/\hbar S_H \mapsto -1/\hbar S_H^E$, we set

$\tau_E \equiv i(\tau - t) \Leftrightarrow \tau = -i(\tau_E + t)$ and $d\tau = -i d\tau_E$, so that

$\tau_E|_{\tau=t} = 0$ and $\tau_E|_{\tau=t'} = \hbar\beta$

→ Path integrals may be used in the context of quantum statistical mechanics.

→ Integrating over the ps, we get $\frac{i}{\hbar} S_H[p,q] \mapsto \frac{i}{\hbar} S_L[q]$:

$$Z(\beta) = \int \mathcal{D}q^a \sqrt{\det(2\pi\hbar G)} \exp\left\{\frac{-1}{\hbar} S_L^E[q]\right\}$$

with $S_L^E = \int_0^{\hbar\beta} d\tau_E \left\{\frac{1}{2} g_{ab} \dot{q}^a \dot{q}^b + V(q)\right\}$

and $G^{ab}(\tau,\tau') = g^{ab}(q(\tau))\,\delta(\tau,\tau')$

We supposed an hamiltonian of form $H = \frac{1}{2} g_{ab} p_a p_b + V(q)$, with $g^{ab}(q)$ a positive defined matrix.

2.3 Semi-classical expansion of $Z(\beta)$

DEF | The semi-classical expansion correspond to the limit where all quantum number are large w.r.t. $\hbar$. Or, formally:

$$\hbar \to 0$$

→ Starting from $Z(\beta) = \int_{\substack{q(0)=q(\hbar\beta)\\ p(0)=p(\hbar\beta)}} \prod_{a,\tau} dq^a(\tau) \prod_{b,\tau'} \frac{dp^b(\tau')}{2\pi\hbar} \exp\left\{-\frac{1}{\hbar} \underbrace{\int_0^{\hbar\beta} d\tau \left(-i\dot{q}^a p_a + H\right)}_{S_E^H}\right\}$

with $H = \frac{1}{2} g^{ab} p_a p_b + V(q)$, we perform $\tau \mapsto \tau/\hbar$, so that:

$q(0) = q(\beta)$, $p(0) = p(\beta)$ and

$$\frac{1}{\hbar} S_H^E = \int_0^{\hbar\beta} d\tau \left(-i\dot{q}^a p_a + H\right) \mapsto \int_0^{\beta} d\tau \left(-i\dot{q}^a p_a/\hbar + H\right)$$

↳ The classical solution is determined by:

$$\delta_p S_H^E = 0 \iff \boxed{-i\dot{q}^a + \hbar\, \partial H/\partial p_a = 0}$$

$$\delta_q S_H^E = 0 \iff \boxed{+i\dot{p}_a + \hbar\, \partial H/\partial q^a = 0}$$

⇒ When $\hbar \to 0$, it's all constant path $q^a = q_0^a$, $p_a = p_a^0$ satisfying the boundary conditions: all constant paths are periodic.

→ Since this is not a unique classical solution, it is better to compute

$$\langle q | e^{-\beta \hat{H}} | p \rangle = \int_{\substack{p(0)=p\\ q(\hbar\beta)=q}} \prod_{a,\tau} dq^a(\tau) \prod_{b,\varsigma} \frac{dp^b(\varsigma)}{2\pi\hbar} \exp\left\{-1/\hbar\, S_H'^E\right\}$$

where $1/\hbar\, S_H'^E = 1/\hbar \int_0^{\hbar\beta} d\tau \underbrace{\left\{\left(-i\dot{q}^a p_a + i q^a \dot{p}_a\right) + H\right\}}_{\text{kinetical term}} + 1/\hbar \underbrace{\frac{1}{2i}\left(q^a(\hbar\beta) p_a(\hbar\beta) + q^a(0) p_a(0)\right)}_{\text{boundary term}}$

↳ In this case $\exists!$ classical solution and $\frac{1}{\hbar} S_H'^E = \beta H + \frac{1}{i\hbar} q^a p_a$

→ A leading order in $\hbar$, one has
$\langle q | e^{-\beta \hat{H}} | p \rangle = e^{-\beta H(q,p)} e^{-q^a p_a / i\hbar} \cdot \frac{1}{(2\pi\hbar)^{n/2}} \equiv C$

while $Z(\beta)_{class} = \int \prod_a dq^a \prod_b dp^b \langle q | e^{-\beta \hat{H}} | p \rangle \langle p | q \rangle$

$$= \int \prod_a dq^a \prod_b \frac{dp^b}{2\pi\hbar} \exp\left\{ -\beta H(q,p) - \frac{1}{i\hbar} q^a p_a + \frac{1}{i\hbar} q^a p_a \right\}$$

PROP | $Z_{class}(\beta) = \int \prod_a dq^a \prod_b \frac{dp_b}{2\pi\hbar} \exp\{-\beta H(q,p)\}$

⊙ Correction in $\hbar$

→ Consider $\langle q | e^{-\beta \hat{H}} | p \rangle$ after having performed the path integral over the p's. The prefactor is: $\left(\frac{g}{2\pi\hbar^2 \beta/(N+1)} \right)^{\frac{N+1}{2}} \equiv C'$ so that:

$$\langle q | e^{-\beta \hat{H}} | p \rangle = C' \int_{\substack{q(0)=q \\ p(\hbar\beta)=p}} \prod_{a,\tau} dq^a(\tau) \exp\left\{ -\frac{1}{\hbar} \int_0^{\hbar\beta} d\tau \left(\frac{1}{2} g_{ab} \dot{q}^a \dot{q}^b + V(q) \right) \right\}$$

$$= C' \int_{\dots} \prod dq^a(\tau) \exp\left\{ \int_0^\beta d\tau \left(\frac{1}{2} \frac{1}{\hbar^2} g_{ab} \dot{q}^a \dot{q}^b + V(q) \right) \right\}$$

→ We decompose $q^a(\tau)$ around its classical solution: $q^a(\tau) = q^a + \tilde{q}^a(\tau)$, the integrate on the fluctuations $\tilde{q}^a$

$$\models C' \int_{\tilde{q}(0)=0=\tilde{q}(\beta)} \prod_{a,\tau} d\tilde{q}^a(\tau) \exp\left\{ -\int_0^\beta d\tau \left(\frac{1}{2} g_{ab} \frac{\dot{\tilde{q}}^a \dot{\tilde{q}}^b}{\hbar^2} + V(q) + \left.\frac{\partial V}{\partial q^a}\right|_q \tilde{q}^a + \frac{1}{2} \left.\frac{\partial^2 V}{\partial q^a \partial q^b}\right|_q \tilde{q}^a \tilde{q}^b \right) \right.$$
$$+ \mathcal{O}(\tilde{q}^3)$$

$$= C' C'' e^{-\beta V(q)} \left(\langle 1 \rangle - \left.\frac{\partial V}{\partial q^a}\right|_q \langle \int_0^\beta d\tau \, \tilde{q}^a(\tau) \rangle \right.$$
$$- \frac{1}{2} \left.\frac{\partial^2 V}{\partial q^a \partial q^b}\right|_q \langle \int_0^\beta d\tau \, \tilde{q}^a(\tau) \tilde{q}^b(\tau) \rangle$$
$$\left. + \frac{1}{2} \left.\frac{\partial V}{\partial q^a}\right|_q \left.\frac{\partial V}{\partial q^b}\right|_q \langle \int_0^\beta d\tau \int_0^\beta d\tau' \, \tilde{q}^a(\tau) \tilde{q}^b(\tau') \rangle + \mathcal{O}(\tilde{q}^3) \right)$$

DEF | The Gaussian approximated expectation value $\langle \; \rangle$ is defined as:
$$\langle X \rangle = \frac{1}{C''} \int \prod_{a,\tau} d\tilde{q}^a(\tau) \, X \exp\left\{ -\int_0^\beta d\tau \, \frac{1}{2} \frac{g_{ab}}{\hbar^2} \dot{\tilde{q}}^a \dot{\tilde{q}}^b \right.$$

↳ To be able to compute it, we need to introduce a new tool.

DEF Given a source b, we introduce the generating functional I_p:

$$I_p[\beta, b] = \int_{\tilde{q}(0)=0=\tilde{q}(\beta)} \prod_{\tau} d\tilde{q}(\tau) \exp\left\{\int_0^\beta -\frac{m^2}{2\hbar^2}\dot{\tilde{q}}^2 + b(\tau)\tilde{q}(\tau)\right\}$$

→ We can write

$$I_p(\beta, b) = \mathcal{Z}'' \langle \exp\{\int_0^\beta b(\tau)\tilde{q}(\tau)\}\rangle$$

$$= \mathcal{Z}'' \exp\{\frac{\hbar^2}{2m}\int_0^\beta d\tau\, d\tau' \,(\underbrace{b(\tau)\Delta(\tau-\tau')b(\tau')}_{\text{propagator}}\}$$

Indeed,

$$I(A, b) = \int \prod_i dx_i \exp\{\frac{-1}{2}A_{ij}x_i x_j + b_i x_i\} = \left(\det\frac{A}{2\pi}\right)^{-1/2}\exp\{\frac{1}{2}b(A^{-1})b\}$$

? We want to inverse the kinetic operator; namely find the Green function associated. We solve: $-\ddot{\Delta}(t, u) = \delta(t-u)$

with B.C. $\Delta(\beta, u) = 0 = \Delta(0, u)$

→ We know that $\dot{\Theta}(t) = \delta(t)$ and we can write

$\Theta(t) = \frac{1}{2}(|\dot{t}| + 1)$ indeed, for $\begin{cases} t>0: 1/2(1+1) = 1 \rightsquigarrow \Theta(t) \\ t<0: 1/2(-1+1) = 0 \end{cases}$

→ The solution is $\Delta(t, u) = -\frac{1}{2}|t-u| + \frac{1}{2}(t+u-2ut/\beta)$

$$= \begin{cases} u - ut/\beta & \text{for } t>u \\ t - ut/\beta & \text{for } t<u \end{cases}$$

→ Let $V=0$. Then $H = p^2/2m$ and

$$\langle q=0|e^{-\beta/2m}|q=0\rangle = \int dp \langle 0|e^{-\beta/2m}|p\rangle\langle p|0\rangle = \mathcal{Z}'\mathcal{Z}''$$

$$= \frac{1}{2\pi\hbar}\sqrt{\frac{2\pi m}{\beta}} = \sqrt{\frac{m}{2\pi\beta\hbar}} \quad \text{so we can write}$$

$$\langle q|e^{-\beta H}|q\rangle = \sqrt{\frac{m}{2\pi\hbar^2\beta}}\, e^{-\beta V(q)}\Big(1 - \frac{1}{2}V''(q)\frac{\hbar^2}{m}\int_0^\beta d\tau\, \Delta(\tau, \tau')$$

$$+ \frac{1}{2}(V'(q))^2\frac{\hbar^2}{m}\int_0^\beta d\tau\int_0^\beta d\tau'\Delta(\tau, \tau') + \mathcal{O}(\hbar^3)\Big)$$

? ↳ The 1-point function vanishes.

→ Let's compute $\int_0^\beta \Delta(\tau, \tau) = \int_0^\beta(\tau - \tau^2/\beta)d\tau = \frac{1}{2}\beta^2 - \frac{1}{3}\beta^3/\beta = \beta^2/6$

and $\int_0^\beta d\tau \int_0^\beta d\tau' \Delta(\tau, \tau') = \int_\tau^\beta d\tau \int_0^\tau d\tau'(\tau' - \tau'\tau/\beta) + \int_0^\beta d\tau \int_\tau^\beta d\tau'(\tau - \tau\tau'/\beta)$

$= \beta^3/12$

So that $Z(\beta) = \sqrt{\frac{m}{2\pi\hbar^2\beta}}\int_{-\infty}^{\infty} dq\, e^{-\beta V}\left(1 + \frac{\hbar^2\beta^3}{24m}(V')^2 - \frac{\hbar^2\beta^2}{12m}V'' + \mathcal{O}(\hbar^3)\right)$

Now, $e^{-\beta V}(V')^2 = \frac{d}{dq}(e^{-\beta V}) \cdot V' \cdot (-1/\beta) = \underbrace{\frac{d}{dq}\left(e^{-\beta V} V' \cdot (-1/\beta)\right)}_{\to 0 \text{ (B. term)}} + \frac{1}{\beta} e^{-\beta V} V''$

so that $Z(\beta) = \sqrt{\frac{m}{2\pi\hbar^2\beta}} \int_{-\infty}^{\infty} dq\, e^{-\beta V}\left(1 - \frac{\hbar^2\beta^2}{24m} V''\right) + O(\hbar^3)$

⊙ Good old harmonic oscillator:

→ We consider $V = \frac{1}{2}kq^2$ with $\omega^2 = k/m$, so that

$$Z(\beta) = \sqrt{\frac{m}{2\pi\hbar^2\beta}} \int_{-\infty}^{\infty} dq \exp\{-\beta k q^2/2\} \left(1 - \frac{\hbar^2\beta^2\omega^2}{24} + O(\hbar^3)\right)$$

$$= \sqrt{\frac{m}{2\pi\hbar^2\beta}} \sqrt{\frac{2\pi}{\beta k}} \left(1 - \frac{\hbar^2\beta^2\omega^2}{24}\right) = \frac{1}{\hbar\beta\omega}(1 - \hbar^2\beta^2\omega^2/24)$$

PROP We found the semi classical approx of the partition fct of the HO:

$$Z(\beta) = \frac{1}{\hbar\beta\omega} - \frac{\hbar\beta\omega}{24}$$

The exact solution is

$$Z(\beta) = \frac{1}{e^{\beta\hbar\omega/2} - e^{-\beta\hbar\omega/2}} = (2\sinh(\hbar\beta\omega/2))^{-1}$$

2.4 Holomorphic representation

2.4.1 Coherent states:

DEF For a single dof (q,p), we define the annihilation (operator) a as

$$a \equiv \frac{1}{\sqrt{2\hbar}}\left(\sqrt{\omega}\, q + i\frac{1}{\sqrt{\omega}} p\right)$$

→ For a HO, $H = p^2/2 + \omega^2 q^2/2 = \frac{1}{2}\hbar\omega(a^* a + a a^*)$ where we used $q = \sqrt{\frac{\hbar}{2\omega}}(a + a^*)$ and $p = -i\sqrt{\frac{\hbar\omega}{2}}(a - a^*)$

↳ The Poisson brackets are $\{q,p\} = 1$, $\{q,q\} = \{p,p\} = 0$, $\{a,a^*\} = -\frac{i}{\hbar}$
Quantifying, we impose $[\hat{q},\hat{p}] = i\hbar$, giving $[\hat{a},\hat{a}^+] = 1$

→ The eigenstates of this representation are:
$\hat{a}|0\rangle$ and $|n\rangle = \frac{(\hat{a}^+)^n}{\sqrt{n!}}|0\rangle$

DEF The coherent states are defined as
$|a\rangle = e^{a\hat{a}^\dagger}|0\rangle$ such that $\hat{a}|a\rangle = a|a\rangle$, $a \in \mathbb{C}$

→ The conjugate state associated is $\langle a^*| = \langle 0| e^{\hat{a}a^*}$
such that $\langle a^*|\hat{a}^\dagger = \langle a^*|a^*$. Then, $\langle b^*|a\rangle = e^{b^* a}$

→ Usually, we had $\hat{q}|q\rangle = q|q\rangle$, $\hat{p}|p\rangle = p|p\rangle$, $\langle q'|q\rangle = \delta(q-q')$
and $\mathbb{1} = \int dq\, |q\rangle\langle q|$
Or, $\langle m|n\rangle = \delta_{mn}$, and $\langle q|\psi\rangle = \psi(q)$

→ We can compute the wave function of the orthonormal state $|n\rangle$ in the basis of coherent states:

PROP $\langle a^*|n\rangle = \langle 0| e^{\hat{a}\hat{a}^\dagger} \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle = \frac{1}{\sqrt{n!}}(a^*)^n = \langle a^*|n\rangle$

? DEMO Assuming $\langle a^*|0\rangle = 1$, an assuming it holds for $n-1$, we have
$\langle a^*|n\rangle = \langle 0| e^{a^*\hat{a}} \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle = \langle 0|[e^{a^*\hat{a}}, \frac{1}{\sqrt{n}}a^\dagger] \frac{1}{\sqrt{(n-1)!}}(a^\dagger)^{n-1}|0\rangle$

$= \frac{1}{\sqrt{n}} a^* \langle a^*|n-1\rangle$

→ Similarly: $\langle n|a\rangle = \frac{1}{\sqrt{n!}} a^n$

→ Going further by writing $\mathcal{H} \ni |\psi\rangle = \sum_n \psi_n |n\rangle$ (hence the name: holonomic basis, we develop in power of a, a^*), we get:
$\psi(a^*) = \langle a^*|\psi\rangle$ and $\bar{\psi}(a) = \langle \psi|a\rangle$

PROP The completeness relation in the coherent state basis is given by $\mathbb{1} = \int \frac{da^*\, da}{2\pi i} e^{-a^* a} |a\rangle\langle a^*|$

Equivalently, we have $\langle \psi|\phi\rangle = \int \frac{da^*\, da}{2\pi i} e^{-a^* a} \bar{\psi}(a)\, \phi(a^*)$

DEMO Let show it for the basis elements:
$\langle n|m\rangle = \underbrace{\int \frac{da^*\, da}{2\pi i}}_{\text{let's figure this complex integral out.}} e^{-a^* a} \frac{1}{\sqrt{n!}}(a)^n \frac{1}{\sqrt{m!}}(a^*)^m = \delta_{nm}$

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We had $a = (\sqrt{\omega}\, q + i\sqrt{\omega^{-1}}\, p)/\sqrt{2\hbar}$. Let's compute the jacobian to perform $\int da^* da \to \int dq\, dp\, |\det J|$:

$$\begin{vmatrix} \partial a^*/\partial q & \partial a^*/\partial p \\ \partial a/\partial q & \partial a/\partial p \end{vmatrix} = \begin{vmatrix} \sqrt{\omega/2\hbar} & -i\sqrt{1/2\omega\hbar} \\ \sqrt{\omega/2\hbar} & i\sqrt{1/2\omega\hbar} \end{vmatrix} = i\hbar$$

We find $\langle n|m\rangle = \int \frac{dp\,dq}{2\pi\hbar} \exp\left\{\frac{-1}{2\hbar}(\omega q^2 + p^2/\omega)\right\} \frac{1}{\sqrt{n!}} \frac{1}{\sqrt{m!}} \left(\frac{\sqrt{\omega}q + i\sqrt{\omega^{-1}}p}{\sqrt{2\hbar}}\right)^n \left(\frac{\sqrt{\omega}.q - i\sqrt{\omega^{-1}}p}{\sqrt{2\hbar}}\right)^m$

$= \delta_{mn}$ (to show)

Now, we want to compute

$$I(\alpha, \alpha^*) = \int \frac{dq\,dp}{2\pi\hbar} \underbrace{\exp\left\{\frac{-1}{2\hbar}\left(\omega q^2 + \frac{1}{\omega}p^2\right) + \alpha^* \frac{\sqrt{\omega}q + i\sqrt{\omega^{-1}}p}{\sqrt{2\hbar}} + \alpha \frac{\sqrt{\omega}q - i\sqrt{\omega^{-1}}p}{\sqrt{2\hbar}}\right\}}_{= \exp\left\{-\frac{1}{2}A_{ij}x^i x^j + b_i x^i + c\right\}}$$

with $A = \begin{pmatrix} \omega/\hbar & 0 \\ 0 & 1/\omega\hbar \end{pmatrix}$

so that $(\det A/2\pi)^{-1/2} = 2\pi\hbar$

Now, we find the extremum of the $e^{-\frac{1}{2}x^TAx + x^Tb + c}$ by acting on it with $\partial_a, \partial_{a^*}$ and imposing $\stackrel{!}{=} 0$. We find $e^{\text{extremum}} = e^{\alpha^*\alpha}$ so that

$$\langle n|m\rangle = \frac{1}{\sqrt{n!}}\frac{1}{\sqrt{m!}} (\partial_\alpha)^m (\partial_{\alpha^*})^n \frac{I(\alpha, \alpha^*)}{e^{\alpha\alpha^*}}\Big|_{\alpha=0, \alpha^*=0}$$

$$= \frac{1}{\sqrt{n!}} (\partial_\alpha)^m \left(\frac{1}{\sqrt{m!}} (\alpha^m e^{\alpha^*\alpha})\right)\Big|_{\alpha=0; \alpha^*=0} = \delta_{mn}$$

→ To remember: $I(\alpha, \alpha^*) = e^{\alpha^*\alpha} = \int \frac{da^* da}{2\pi i} \exp\{-a^*a + \alpha^* a + \alpha a^*\}$

2.4.2 Kernel and normal symbol:

→ Given an operator $\hat{O}$, its matrix elements in an orthonormal basis $\{|n\rangle\}$ are $O_{mn} = \langle n|\hat{O}|m\rangle$. We can write // $A(e_i) = e_j A^j{}_i$

$$\hat{O} = \sum_{n,m} |n\rangle O_{nm} \langle m|$$

DEF The kernel of $\hat{O}$ in the holomorphic representation is

$$O(a^*, a) \equiv \langle a^*|\hat{O}|a\rangle$$

↳ We can write $O(a^*, a) = \sum_{n,m} \langle a^*|n\rangle O_{nm} \langle m|a\rangle$

$$= \sum_{n,m} \frac{(a^*)^n}{\sqrt{n!}} O_{nm} \frac{(a)^m}{\sqrt{m!}}$$

→ We have $(\hat{O}|\psi\rangle)(a^*) = \langle a^*|\hat{O}|\psi\rangle = \int d\alpha^* d\alpha/(2\pi i)\, . \langle a^*|O\, e^{-\alpha^*\alpha}|\alpha\rangle\langle\alpha^*|\psi\rangle$

$$= \int \frac{d\alpha^* d\alpha}{2\pi i} . e^{-\alpha^*\alpha}\, O(a^*, \alpha)\, \psi(\alpha^*)$$

Similarly, $(\hat{O}_1\hat{O}_2)(a^*, a) \equiv \langle a^*|\hat{O}_1\hat{O}_2|a\rangle$
$= \int \frac{d\alpha^* d\alpha}{2\pi i} O_1(a^*, \alpha)\, O_2(\alpha^*, a)\, e^{-\alpha^*\alpha}$ } kernel associated to $\hat{O}_1\hat{O}_2$

Furthermore $\hat{a}^+$ and $\hat{a}$ act like multiplication and differentiation by a^* in the holomorph rep.:

$$(\hat{a}^+|\psi\rangle)(a^*) \equiv \langle a^*|\hat{a}^+|\psi\rangle = a^*\psi(a^*)$$

? $$(\hat{a}|\psi\rangle)(a^*) \equiv \langle a^*|\hat{a}|\psi\rangle = \frac{\partial}{\partial a^*}\psi(a^*)$$

DEF The **normal symbol** of an operator $\hat{O}(\hat{a}, \hat{a}^+)$ is $O^N(a^*, a)$, a classical function that encodes all the info of the quantum one.
To $\hat{O}(\hat{a}, \hat{a}^+) \mapsto O^N(a, a^*)$, one orders $\overleftarrow{\hat{a}^+}; \overrightarrow{\hat{a}}$ using commutation relations, then replace $\hat{a}^+ \mapsto a^*$ and $\hat{a} \mapsto a$.

ex: $\hat{a}^+\hat{a} \mapsto a^*a$
$\hat{a}\hat{a}^+ = [\hat{a}, \hat{a}^+] + \hat{a}^+\hat{a} = 1 + a^*a$

→ ⚠ Not the same that the normal ordered operator $: \;:$
$:\hat{a}^+\hat{a}: = \hat{a}^+\hat{a}$ and $:\hat{a}\hat{a}^+: = :\hat{a}^+\hat{a}:$ (no commutation!)
The normal order op. erase the initial order.

PROP The projector on the vacuum can be represented as

$$|0\rangle\langle 0| = :e^{-\hat{a}^+\hat{a}}:$$

$\boxed{\text{DEMO}}$ Without loss of generality, we consider the state $|n\rangle$. Then we want to show that $|0\rangle\langle 0|n\rangle = |0\rangle \Leftrightarrow :e^{-\hat{a}^+\hat{a}}:|n\rangle = |0\rangle = \delta_{n0}|n\rangle$.
Equivalently, we can prove $[:e^{-\hat{a}^+\hat{a}}:, (\hat{a}^+)^n] = -(\hat{a}^+)^n :e^{-\hat{a}^+\hat{a}}:$
→ $n=1$: $[\Sigma(-1)^n(\hat{a}^+)^n(\hat{a})^n/n!, \hat{a}^+] = -\hat{a}^+ :e^{-\hat{a}^+\hat{a}}:$
→ $n>1$: $[:e^{-\hat{a}^+\hat{a}}:, (\hat{a}^+)^n] = \hat{a}^+[:e^{-\hat{a}^+\hat{a}}:, (\hat{a}^+)^{n-1}] + [:e^{-\hat{a}^+\hat{a}}:, \hat{a}^+](\hat{a}^+)^{n-1}$
$= -(\hat{a}^+)^n :e^{-\hat{a}^+\hat{a}}: - \hat{a}^+ :e^{-\hat{a}^+\hat{a}}: (\hat{a}^+)^{n-1}$
$= -(\hat{a}^+)^n :e^{-\hat{a}^+\hat{a}}: - \hat{a}^+[:e^{-\hat{a}^+\hat{a}}:, (\hat{a}^+)^{n-1}] - (\hat{a}^+)^n :e^{-\hat{a}^+\hat{a}}:$
$= -(\hat{a}^+)^n :e^{-\hat{a}^+\hat{a}}$ ▨

PROP The kernel and the normal symbol of an operator are related by

$$O(a^*, a) = e^{a^*a}\, O^N(a^*, a)$$

DEMO Indeed: $\hat{O} = \sum_{n,m} O_{nm} \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle\langle 0| \frac{(\hat{a})^m}{\sqrt{m!}}$ with $|0\rangle\langle 0| = :e^{-\hat{a}^\dagger\hat{a}}:$

$$= \sum_{n,m} O_{nm} : \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} e^{-\hat{a}^\dagger \hat{a}} \frac{(\hat{a})^m}{\sqrt{m!}} :$$ and since $O(a^*,a) = \sum_{n,m} O_{nm} \frac{(a^*)^n}{\sqrt{n!}} \frac{a^m}{\sqrt{m!}}$,

$$O^N(a^*,a) = \sum_{n,m} O_{nm} \frac{(a^*)^n}{\sqrt{n!}} e^{-a^*a} \frac{a^m}{\sqrt{m!}} = O(a^*,a)\, e^{-a^*a}$$ ▨

→ The normal symbol of an op. in the normal order is the op. itself.

→ In the holomorphic rep., the trace is given by:

$$\mathrm{Tr}\,\hat{O} = \sum_n \langle n|\hat{O}|n\rangle = \int \frac{da^* da}{2\pi i} \sum_n \langle n|a\rangle \langle a^*| \hat{O} |n\rangle e^{-a^*a}$$

$$= \int \frac{da^* da}{2\pi i} O(a^*,a) e^{-a^*a}$$

2.4.3 Evolution operator in holomorphic representation. ($\hbar = 1$)

DEF The evolution op. in the holomorphic rep. $U(a^*,t';a,t)$ is

$$U(a^*,t';a,t) = \langle a^*;t'|a;t\rangle = \langle a^*| e^{-i\hat{H}(t'-t)}|a\rangle$$

→ For $t'-t = \epsilon \ll 1$, we have

$$\langle a^*|1 - i\epsilon\hat{H}|a\rangle = e^{a^*a} - i\epsilon\mathcal{H}(a^*,a) = e^{a^*a}(1 - i\epsilon \underbrace{h(a^*,a)}_{\text{normal symbol}})$$

$$= e^{a^*a - i\epsilon h(a^*,a)}$$

where $\mathcal{H}(a^*,a)$ is the kernel of the Hamiltonian $\hat{H}$ while $h(a^*,a)$ ↑.

→ For $(t'-t)/N = \epsilon$, $U(a^*,t';a,t) = \langle a^*| \underbrace{e^{-i\epsilon\hat{H}} \cdots e^{-i\epsilon H}}_{N \text{ times}} |a\rangle$ and inserting $\mathbb{1} = \int da^* da/(2\pi i) \cdot e^{-a^*a} |a\rangle\langle a^*|$;

$$U = \int \frac{da^*_{N-1} da_{N-1}}{2\pi i} e^{-a^*_{N-1}a_{N-1}} |a_{N-1};z_{N-1}\rangle\langle a^*_{N-1}, z_{N-1}| \cdots \int \frac{da^*_1 da_1}{2\pi i} e^{-a^*_1 a_1} |a_1;z_1\rangle\langle a^*_1;z_1|$$

$$= \lim_{N\to\infty} \int \prod_{k=1}^{N-1} \frac{da^*_k da_k}{2\pi i} e^{S'_D}$$ with $S'_D = -i\epsilon(h(a^*_N, a_{N-1}) + \dots + h(a^*_1, a_0))$
$+ a^*_N a_{N-1} + \dots + a^*_1 a_0 - a^*_{N-1}a_{N-1} - \dots - a^*_1 a_1$

Now, we take the limit to $N \to \infty$ (continuity).

PROP The evolution can be rewritten as:

$$U(a^*,t';a,t)=\int_{a(t)=a}^{a^*(t')=a^*}\prod_\tau \frac{da^*(\tau)\,da(\tau)}{2\pi i}\exp\{iS_H'[a^*,a]\}$$

with S_H' the Holomorphic Action given by

$$S_H'=\int_t^{t'}d\tau\Big(\underbrace{\frac{1}{2i}(\dot a^*a-a^*\dot a)}_{\text{symplectic term}}-\underbrace{h(a^*,a)}_{\text{hamiltonian term}}\Big)+\underbrace{\frac{1}{2i}\Big(a^*(t')a(t')+a^*(t)a(t)\Big)}_{\text{boundary term}}$$

$\boxed{\text{DEMO}}$ Indeed, let's discretize iS_H'; one finds:

$$\frac{1}{2}\Big[\varepsilon\cdot\frac{a_N^*-a_{N-1}^*}{\varepsilon}\cdot a_{N-1}+\dots+(a_1^*-a_0^*)a_0-a_N^*(a_N-a_{N-1})-\dots-a_1^*(a_1-a_0)+a_N^*a_N+a_0^*a_0\Big]$$

$$-i\varepsilon\big[h(a_N^*,a_{N-1})+\dots+h(a_1^*,a_0)\big]$$

→ Without performing the full computation each time, we can follow the following rule to know which action use in the path integral.

RULE The action to be used in the path integral is the one that has a true extremum when taking into account the boundary conditions $\delta a^*(t')=0=\delta a(t)$.

Indeed, for $\langle q',t'|q,t\rangle$ we had $S_H=\int d\tau(\dot q p-H)$, then

$$\delta S_H=\int d\tau\Big[\underbrace{\delta p\Big(\dot q-\frac{\delta H}{\delta p}\Big)}_{\text{Hamilton eq.}}+\underbrace{\frac{d}{dt}(\dot q p)}_{\text{boundary}}+\underbrace{\delta q\Big(-\dot p-\frac{\delta H}{\delta q}\Big)}_{\text{Hamilton eq.}}\Big]$$

→ Varying S_H', we get:

$$\delta S_H'=\int_t^{t'}d\tau\Big\{\underbrace{\Big(\frac{1}{i}\dot a^*-\frac{\partial h}{\partial a}\Big)}_{\text{EOM}}\delta a+\underbrace{\Big(\frac{-1}{i}\dot a-\frac{\partial h}{\partial a^*}\Big)}_{\text{EOM}}\delta a^*+\frac{1}{2i}\Big[\delta a^* a\Big]_t^{t'}-\frac{1}{2i}\Big[a^*\delta a\Big]_t^{t'}$$

$$+\frac{1}{2i}\Big(\delta a^*(t')a(t')+a^*(t')\delta a(t')+\delta a^*(t)a(t)+a^*(t)\delta a(t)\Big)\Big\}$$

On the EOM, with B.C. $\delta(a^*(t'))=0$ and $\delta(a(t))=0$, we're left with

$$\delta S_H'=\frac{-1}{2i}\delta a^*(t)a(t)-\frac{1}{2i}a^*(t')\delta a(t')+\frac{1}{2i}a^*(t')\delta a(t')+\frac{1}{2i}\delta a^*(t)a(t)=0$$

2.4.4 Study case: forced harmonic oscillator:

→ We want to compute $\langle a^*, t' | a, t \rangle$ with the path integral for the forced HO. We consider:

$$\hat{H} = \omega\left(\hat{a}^\dagger \hat{a} + \frac{\alpha}{2}\right) - j(t)\,\hat{a}^\dagger - j^*(t)\,\hat{a} \qquad (\alpha = 1)$$

$\alpha = 1 \Leftrightarrow$ symmetrical ordering during quantization

The corresponding normal symbol is:

$$h(a^*, a) = \omega(a^* a + \alpha/2) - j(t)\, a^* - j^*(t)\, a$$

→ To compute the path integral, since all discretized integral are gaussian, we admit the shortcut that the result $\sim e^{\text{extremum}}$ without prefactor. The Hamilton equations are:

$$\begin{cases} \dot{a} = -i \dfrac{\delta h}{\delta a^*} = -i(\omega a - j) & \text{with B.C. } a(t) = a \\ \dot{a}^* = i \dfrac{\delta h}{\delta a} = i(\omega a^* - j^*) & \quad " \quad " \quad a^*(t') = a^* \end{cases}$$

→ Notice that the B.C. are not conjugated. At the boundary, a and a^* are 2 independant variables.

→ We solve the equations by 1st solving it without source, then injecting the 1st solution in the whole system. Constants are determined by B.C.:

$$\begin{cases} a(\tau) = e^{-i\omega(\tau - t)}\, a + i \int_t^\tau d\tau'\, j(\tau')\, e^{-i\omega(\tau - \tau')} \\ a^*(\tau) = e^{-i\omega(t' - \tau)}\, a^* + i \int_\tau^{t'} d\tau'\, j^*(\tau')\, e^{-i\omega(\tau' - \tau)} \end{cases}$$

→ Recall that $\partial_x \int^x dy\, f(y) = f(x)$

↳ Substituting this solution in the classical action, we get:

$$\ln U(a^*, t'; a, t) = -i(t' - t)\frac{\alpha\omega}{2} + a^* e^{-i\omega(t' - t)}\, a$$
$$+ i a^* \int_t^{t'} d\tau\, \frac{j(\tau)}{\hbar}\, e^{-i\omega(t' - \tau)} + i a \int_t^{t'} d\tau\, \frac{j^*(\tau)}{\hbar}\, e^{-i\omega(\tau - t)}$$
$$- \int_t^{t'} d\tau \int_t^{t'} d\tau'\, \frac{j^*(\tau)}{\hbar}\, e^{-i\omega(\tau - \tau')}\, \theta(\tau - \tau')\, \frac{j(\tau')}{\hbar}$$

Usefull for the reduction formulas and thermal field theory

2.5 Reduction formulas

2.5.1 S-matrix of forced harmonic oscillator:

DEF The $\hat{S}$ operator is given by
$\hat{S} = \lim_{\substack{t' \to +\infty \\ t \to -\infty}} \hat{S}(t',t)$ with $\hat{S}(t',t) = e^{i\hat{H}_0 t'} \hat{U}(t',t) e^{-i\hat{H}_0 t}$

PROP The kernel of $\hat{S}(t',t)$ in the holomorphic representation is
$S(a^*,t';a,t) = \langle a^*,t' | e^{i\hat{H}_0 t} \mathbb{1} \hat{U}(t',t) \mathbb{1} e^{-i\hat{H}_0 t} | a,t \rangle$

$$= \int \frac{d\alpha^* d\alpha}{2\pi i} \frac{d\beta^* d\beta}{2\pi i} \underbrace{e^{-\alpha^*\alpha} e^{-\beta^*\beta} \langle a^* | e^{i\hat{H}_0 t'} | \alpha \rangle U(\alpha^*,t';\beta,t) \langle \beta^* | e^{-i\hat{H}_0 t} | a \rangle}_{e^A}$$

$$= \int \frac{d\alpha^* d\alpha}{2\pi i} \frac{d\beta^* d\beta}{2\pi i} e^A$$

with $A = -\alpha^*\alpha + a^* e^{i\omega t'} \alpha + \alpha^* e^{-i\omega(t'-t)} \beta + i\alpha^* \int_t^{t'} d\tau\, e^{-i\omega(t'-\tau)} j(\tau)$
$+ i\alpha \int_t^{t'} d\tau\, j^*(\tau) e^{-i\omega(\tau - t)} - \int_t^{t'} d\tau \int_t^{t'} d\tau'\, j^*(\tau) \Theta(\tau-\tau') e^{-i\omega(\tau-\tau')} j(\tau')$
$-\beta^*\beta + \beta^* e^{-i\omega t} a$

Indeed, $\langle a^* | e^{i\hat{H}_0 t'} | \alpha \rangle$ is a particular case of $\ln U(a^*,t';a,t)$ with $j=0$ and $t'-t \mapsto -t'$.

→ Reduction formula ≡ obtain S-matrix element from Green functions.

→ Formal result: $\ln S(a^*,t';a,t) = a^* a + i a^* \int_t^{t'} d\tau\, j(\tau) e^{+i\omega\tau}$

$$+ i a \int_t^{t'} d\tau\, j^*(\tau) e^{i\omega\tau} - \int_t^{t'} d\tau \int_t^{t'} d\tau\, j^*(\tau) e^{-i\omega(\tau-\tau')} \Theta(\tau-\tau') j(\tau')$$

It follows that the normal symbol is
$\ln S^N(a^*,t';a,t) = \ln S(a^*,t';a,t) - a^* a$

2.5.2 S-matrix of interacting scalar field:

→ In Fourier transform, the scalar field $\hat{\phi}(x)$ with real external source $j(x)$ is a superposition of decoupled harmonic oscillators:

→ $\hat{H}_0^{j} = \int d^3x \left\{ \frac{1}{2}\hat{\pi}^2 + \frac{1}{2}\partial_k\hat{\phi}\,\partial^k\hat{\phi} + \frac{1}{2}m^2\hat{\phi}^2 - j\hat{\phi} \right\}$; where:

↳ $j(\vec{x}) = (2\pi)^{-3/2}\int d^3k \sqrt{2\omega(\vec{k})}\, e^{i\vec{k}\cdot\vec{x}}\,\tilde{j}(\vec{k})$ with $\tilde{j}^*(\vec{k}) = \tilde{j}(-\vec{k})$

↳ $\hat{\phi}(\vec{x}) = (2\pi)^{-3/2}\int \frac{d^3k}{\sqrt{2\omega(\vec{k})}}\left(\hat{a}(\vec{k})\, e^{i\vec{k}\cdot\vec{x}} + \hat{a}^\dagger(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}}\right)$

→ The forced hamiltonian for a scalar field can be written as
$\hat{H}_0^{j} = \int d^3k \left\{ \omega(k)\,\hat{a}^\dagger(k)\hat{a}(k) - \tilde{j}_k\hat{a}^\dagger(k) - \tilde{j}^*_k\hat{a}(k) \right\}$ when $x=0$

→ To use previous development, but for an ∞ number of dof, we need to perform $\int d^3k$ on all oscillators.

→ $$\ln S_0^N(a^*,t';a,t) = \underbrace{i\int_{-\infty}^{\infty} d\tau \int d^3k \left\{ a^*(k)\, e^{i\omega(k)\tau}\,\tilde{j}(\tau,k) + \tilde{j}^*(\tau,k)\, e^{-i\omega(k)\tau}\, a(k) \right\}}_{\circledast}$$
$$\underbrace{-\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' \int d^3k\, \tilde{j}^*(\tau,k)\, e^{-i\omega(k)(\tau-\tau')}\,\theta(\tau-\tau')\,\tilde{j}(\tau',k)}_{\circledast}$$

DEF | We introduce the asymptotic field $\phi_{as}(x)$, defined as
$$\phi_{as}(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3k}{\sqrt{2\omega(k)}}\left\{ a(k)\, e^{ikx} + a^*(k)\, e^{-ikx} \right\}$$

Then, $\circledast = i\int d^4x\, \phi_{as}(x)\, j(x)$

$$\circledast = \frac{i}{2}\int d^4x \int d^4x'\, j(x)\,\Delta_F(x-x')\, j(x')$$

with $\Delta_F = i\langle 0|T\{\hat{\phi}(x)\hat{\phi}(y)\}|0\rangle = \frac{i}{(2\pi)^3}\int \frac{d^3k}{2\omega}\left(e^{ik(\vec{x}-\vec{y})}\,\theta(x^0-y^0) + (x\leftrightarrow y)\right)$

Recall: $\theta(t) = \frac{-1}{2\pi i}\int_{-\infty}^{\infty} ds\, \frac{e^{-ist}}{s+i\epsilon}$

PROP | $S^N(a^*,+\infty;a,\infty) = e^{i\int d^4x\, \phi_{as}(x) j(x)}\, Z_0[j]/Z_0[0]$
with $\frac{Z_0[j]}{Z_0[0]} = \exp\left\{ \frac{i}{2\hbar}\int d^4x\, j(x)\,\Delta_F(x-x')\, j(x') \right\}$

⊙ Remarks:

① $\frac{\delta}{\delta j(y)} S_0^N = \left(i\phi_{as}(y) + i\int d^4x\, \Delta_F(x,y)\, j(x) \right) S_0^N$

② We can obtain $\hat{S}_0$ from S_0^N: we take the normal ordering and replace $\phi_{as} \mapsto \hat{\phi}$:

$$\hat{S}_0 = :\exp\{ i\int d^4x\, \hat{\phi}(x)\, j(x)\}: \frac{Z_0[J]}{Z_0[0]}$$

→ If we add an interaction $I_1[\phi]$, we treat it perturbatively:

$S^N(a^*, \infty; a, -\infty) = e^{i I_1[\frac{1}{i}\frac{\delta}{\delta J}]} S_0^N$

and $\hat{S} = \exp\{ i I_1 \left[\frac{1}{i}\frac{\delta}{\delta j}\right]\} :\exp\{ i\int d^4x\, \hat{\phi}(x)\, j(x)\}: \frac{Z_0[j]}{Z_0[0]}$

→ If no external source wanted, set $j=0$ at the <u>end</u> of the computation

③ Dividing by $Z[0]$ instead of $Z_0[0]$ removes the vacuum part of the diagrams for Green's functions.

④ A generalization of the generating functional of the S-matrix in the presence of a source, where the external states (contained in the Fourier coef. $\tilde{\phi}(x)$) are off-the mass-shell:

$$\tilde{S}^N(\tilde{\phi}, j) = \exp\{ i I_1 \left[\frac{1}{i}\frac{\delta}{\delta j}\right]\} \exp\{ i\int d^4x\, \tilde{\phi}(x)\, j(x)\} \frac{Z_0[j]}{Z_0[0]}$$

Then: $\tilde{S}^N(\phi_{as}, j) = S^N(a^*, t'; a, t)$

and $\tilde{S}^N(0, j) = Z_0[j]/Z_0[0]$

2.5.3 S-matrix from Green's functions:

→ Computing the green's functions give the S matrix elements.

→ Writing S^N as a Taylor serie in ϕ_{as}:

$$S^N = \sum_n \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \, \underbrace{\phi_{as}(x_1) \cdots \phi_{as}(x_n)}_{\text{on shell } (-\Box+m^2)\phi_{as}=0} \, \overbrace{S^N(x_1,\dots,x_n)}^{\text{Taylor coef.}}$$

where $S^N(x_1,\dots,x_n) = \frac{\delta}{i\delta\Phi(x_1)} \cdots \frac{\delta}{i\delta\Phi(x_n)} \tilde{S}^N \Big|_{j=0=\Phi}$

$$= \exp\{i I_1[\tfrac{1}{i}\tfrac{\delta}{\delta j}]\} j(x_1) \cdots j(x_n) \exp\{i \int d^4x \, \Phi(x) j(x)\} \frac{Z_0[j]}{Z_0[0]} \Bigg|_{\substack{j=0 \\ \Phi=0}}$$

→ Recalling that $(-\Box+m^2)\phi_{as} = 0$

$(-\Box+m^2)\Delta_F(x-y) = \delta(x-y)$, we have

$$(-\Box_{x_\ell}+m^2) \frac{\delta}{i\delta j(x_\ell)} Z(j) = \exp\{i I_1[\tfrac{\delta}{\delta j}]\} j(x_\ell) \frac{Z_0[j]}{Z_0[0]}$$

Indeed: $\hookrightarrow = (-\Box_{x_\ell}+m^2) \int d^4x \{\Delta_F(x-y) j(x)\} Z_0[j]/Z_0[0]$

Then, it implies that

$$\int d^4x_1 \cdots d^4x_n \, \phi_{as}(x_1) \cdots \phi_{as}(x_n) \left[S^N(x_1,\dots,x_n) - \left((-\Box_{x_1}+m^2) \frac{\delta}{i\delta j(x_1)} \cdots (-\Box_n+m^2) \frac{\delta}{i\delta j(x_n)} Z[j] \right) \right]_{j=0}$$

$$= 0$$

→ In term of Feynman rules, applying $(-\Box_{x_\ell}+m^2)\frac{\delta}{i\delta j(x_\ell)}$ to $Z[j]$ amputates the external propagator at x_ℓ of the green function.

PROP The normal symbol of the S-matrix is obtained by taking Green's function of order n, amputating the external propagators, multiplying instead by ϕ_{as} at the external points, dividing by $n!$ and summing over n. One gets:

$$S^N = \exp\left\{\int d^4x \, \phi_{as}(x)(-\Box+m^2)\frac{\delta}{\delta j(x)}\right\} \frac{Z[j]}{Z[0]}$$

DEF We can decompose $\hat{S} = \hat{\mathbb{1}} + i\hat{T}$

where $\hat{T}$ is the transition matrix

→ Going to the kernel, we find

$$S(a^*,\infty;a,-\infty) = \langle a^*,\infty|\hat{S}|a,-\infty\rangle$$

$$= \underbrace{\exp\{\int d^3k\, a^*(\vec{k})\, a(\vec{k})\}}_{\hat{\mathbb{1}}} \exp\{\int d^4x\, \phi_{as}(x)(-\Box+m^2)\frac{\delta}{\delta j(x)}\} \frac{Z[j]}{Z[0]}$$

→ For the free theory $I_1[\phi]=0=j$, so that

$$S(a^*,\infty;a,-\infty)\Big|_{I_1=0=j} \equiv S_0(a^*,\infty;a,-\infty) = \exp\{\int d^3k\, a^*(\vec{k})\, a(\vec{k})\}$$

Since $\langle a^*,\infty| = \langle +\infty;0|\exp\{\int d^3k\, \hat{a}(\vec{k},\infty)\, a^*(\vec{k})\}$

$|a,-\infty\rangle = \exp\{\int d^3k\, a(\vec{k})\, \hat{a}^*(\vec{k},-\infty)\}|0;-\infty\rangle$

with $\hat{a}(\vec{k},t) = \hat{a}(\vec{k})e^{-i\omega t}$ and $|0,t\rangle = e^{-i\omega t}|0\rangle$

One can check that $\langle 0|0\rangle = 1$ and that

$$\langle\vec{q}|\vec{p}\rangle = \frac{\delta}{\delta a^*(\vec{q})}\frac{\delta}{\delta a(\vec{p})} S_0(a^*,+\infty;a,-\infty)\Big|_{\substack{a^*=0\\ a=0}} = \delta^3(\vec{q}-\vec{p})$$

→ Now, $\langle a^*,\infty|iT|a,-\infty\rangle = \exp\{\int d^4x\, \phi_{as}(x)(-\Box+m^2)\frac{\delta}{\delta j(x)}\}\frac{Z[j]}{Z[0]}$

This gives:

$$\langle +\infty;0|\prod_i \hat{a}(\vec{q}_i,+\infty)(iT)\prod_j \hat{a}^\dagger(\vec{p}_j,-\infty)|0;-\infty\rangle$$

$$= \prod_i \frac{\delta}{\delta a^*(\vec{q}_i)}\prod_j\frac{\delta}{\delta a(\vec{p}_j)} e^{\int d^4x\, \phi_{as}(x)(-\Box+m^2)\delta/\delta j(x)} \frac{Z[j]}{Z[0]}\Big|_{\substack{a^*=0=a\\ j=0}}$$

$$= \prod_i \int d^4y_i \frac{e^{-iq_iy_i}}{\sqrt{2\omega(\vec{q}_i)}(2\pi)^{3/2}}(-\Box_{y_i}+m^2)\left(\frac{i}{\hbar}\right)$$

$$\times \prod_j \int d^4x_j \frac{e^{ip_jy_j}}{\sqrt{2\omega(\vec{p}_j)}(2\pi)^{3/2}}(-\Box_{x_j}+m^2)\left(\frac{i}{\hbar}\right)$$

? $$\times \frac{\langle +\infty,0|T\{\prod_i\hat{\phi}(y_i)\prod_j\hat{\phi}(x_j)\}|0;-\infty\rangle}{\langle +\infty;0|0;-\infty\rangle}$$

: amputes the external propagator of the states in and out

2.6 Finite temperature results

2.6.1 Harmonic oscillator; Partition function, thermal 2-pt function:

→ We know that $Z(\beta) = \mathrm{Tr}\, e^{-\beta \hat{H}} = \mathrm{Tr}\, \hat{U}(t',t)$ with $t'-t = -i\hbar\beta$. We also found $\ln U(a^*,t';a,t)$ (p19). We can rewrite it as

?
$$U(a^*,a;-i\hbar\beta) = \exp\left\{-\frac{\alpha\hbar\omega\beta}{2} + a^* e^{-\hbar\beta\omega} a\right\}$$

→ Thus, $Z(\beta) = \mathrm{Tr}\, e^{-\beta\hat{H}} = \mathrm{tr}\, \hat{U}(t',t) = \sum_n \langle n|\hat{U}(t',t)|n\rangle$

$$= \int \frac{da^* da}{2\pi i} e^{-a^* a} \sum_n \langle n|a^*\rangle \langle a|\hat{U}(t',t)|n\rangle$$

$$= \int \frac{da^* da}{2\pi i} e^{-\alpha\hbar\omega\beta/2} e^{-a^* a + a^* e^{-\hbar\beta\omega} a}$$

Redefining $a \mapsto \tilde{a}/\sqrt{1-e^{-\hbar\beta\omega}}$ and $a^* = \tilde{a}^*/\sqrt{1-e^{-\hbar\beta\omega}}$,

$\exp\{-a^*(1-e^{-\hbar\beta\omega})a\} = e^{-\tilde{a}^*\tilde{a}}$. Now, $\det[\partial a/\partial\tilde{a}] = (1-e^{-\beta\hbar\omega})^{-1}$

so $Z(\beta) = e^{-\alpha\hbar\omega\beta/2} \cdot \frac{1}{1-e^{-\beta\hbar\omega}} \underbrace{\int \frac{d\tilde{a}^* d\tilde{a}}{2\pi i} e^{-\tilde{a}^*\tilde{a}}}_{=1}$

PROP For $\alpha = 1$, the partition function reads

$$Z(\beta) = \frac{1}{2\sinh(\hbar\beta\omega/2)}$$

⊙ Generalization

→ We put $\begin{cases} t' \mapsto t'-t \\ t \to 0 \end{cases}$ and use $\begin{cases} t'-t = -i\hbar\beta \\ \tau = -i\lambda \\ \tau' = -i\lambda' \end{cases}$. The expression of $\ln U(a^*,a;-i\hbar\beta)$ becomes:

$$\ln U(a^*,a;-i\hbar\beta) = -\frac{\alpha\hbar\beta\omega}{2} + a^* e^{-\hbar\beta\omega} a + a^* \int_0^{\hbar\beta} d\lambda \frac{j(\lambda)}{\hbar} e^{-\omega(\hbar\beta-\lambda)}$$

$$+ a \int_0^{\hbar\beta} d\lambda \frac{j^*(\lambda)}{\hbar} e^{-\omega\lambda} + \int_0^{\hbar\beta} d\lambda \int_0^{\beta} d\lambda' \frac{j^*(\lambda)}{\hbar} e^{-\omega(\lambda-\lambda')} \theta(\lambda-\lambda') \frac{j(\lambda)}{\hbar}$$

→ Taking the trace now gives

$$\ln Z(\beta;j) = \ln Z(\beta;0) + \int_0^{\hbar\beta} d\lambda \int_0^{\hbar\beta} d\lambda \frac{j^*(\lambda)}{\hbar} \frac{j(\lambda')}{\hbar} e^{-\omega(\lambda-\lambda')} \left\{\theta(\lambda-\lambda') + \frac{e^{-\hbar\beta\omega}}{1-e^{-\hbar\beta\omega}}\right\}$$

⊙ Forced harmonic oscillator:

→ For the harmonic oscillator $a = \frac{\sqrt{\omega}\, q + i\sqrt{\omega}^{-1} p}{\sqrt{2}\, \hbar}$ forced by

$$(*)\ \frac{1}{\hbar}\int_0^{\hbar\beta} d\lambda \left\{ f^*(\lambda) a(\lambda) + f(\lambda) a^*(\lambda) \right\}$$ (recall $\lambda = i\tau \rightsquigarrow$ Euclidean)

taking a real coupling $f = \sqrt{\frac{\hbar}{2\omega}}\, J_R$, $J_R \in \mathbb{R}$, the source becomes

$$(*) = \frac{1}{\hbar}\int_0^{\hbar\beta} d\lambda\ J_R(\lambda) q(\lambda)$$ and we can write

$$\ln Z(\beta, J_R) = \ln Z(\beta, 0) + \frac{1}{2\hbar\omega}\int_0^{\hbar\beta} d\lambda \int_0^{\hbar\beta} d\lambda'\ J_R(\lambda) J_R(\lambda') e^{-\omega(\lambda-\lambda')} \left\{ \Theta(\lambda-\lambda') + \frac{e^{-\hbar\beta\omega}}{1-e^{-\hbar\beta\omega}} \right\}$$

Symmetrizing $e^{-\omega(\lambda-\lambda')}\left\{\Theta(\lambda-\lambda') + e^{-\hbar\beta\omega}/(1-e^{-\hbar\beta\omega})\right\}$ and writting $e^{\beta\omega/2} = e^{-\beta\omega/2} + 2\sinh(\beta\omega/2)$, we get

$$\frac{e^{-\beta\omega/2}}{2\sinh(\beta\omega/2)}\left(e^{\omega(\lambda-\lambda')} + e^{-\omega(\lambda-\lambda')}\right) + e^{-\omega|\lambda-\lambda'|}$$

$$= \cosh(\omega|\lambda-\lambda'| - \hbar\beta\omega/2) / 2\sinh(\hbar\beta\omega/2)$$

so that $$\ln Z(\beta, J_R) = \ln Z(\beta, 0) + \frac{1}{2\hbar\omega}\int_0^{\hbar\beta} d\lambda \int_0^{\hbar\beta} d\lambda'\ J_R(\lambda) J_R(\lambda') \frac{\cosh(\omega(|\lambda-\lambda'| - \hbar\beta/2))}{2\sinh(\hbar\beta\omega/2)}$$

⊙ Thermal 2-points function:

DEF The thermal 2-points function $G_\beta(x,y)$ at temperature $T = \beta^{-1}$ is the expectation value of the field operator in the thermal state:

$$G_\beta(x,y) \equiv \langle \Upsilon\, \hat{\Phi}(x)\, \hat{\Phi}(y) \rangle_\beta$$

$$= \frac{\mathrm{Tr}\left\{ e^{-\beta\hat{H}}\, \Upsilon\, \hat{\Phi}(x)\, \hat{\Phi}(y) \right\}}{\mathrm{Tr}\left\{ e^{-\beta\hat{H}} \right\}}$$

→ For instance: $G(\lambda, 0) = Z(\beta)^{-1}\, \mathrm{Tr}\,\hat{q}(\lambda)\, \mathrm{Tr}\,\hat{q}(0)\, e^{-\beta\hat{H}}$

$$= \left[\hbar \frac{\delta}{\delta J_R(\lambda)} \cdot \hbar \frac{\delta}{\delta J_R(0)} \ln Z(\beta; J_R) \right]\Big|_{J_R = 0}$$

$$G(\lambda, 0) = \frac{\hbar}{2\omega} \frac{\cosh(\omega\lambda - \omega\hbar\beta/2)}{\sinh(\hbar\beta\omega/2)}$$

2.6.2 Thermal correlation functions: (2.5.1 Wipf)

→ Thermal correlation functions give acces to the energy and the wave function of the first excited state.

→ Going in the Euclidean ($\tau = it$) in the Heisenberg picture, we write
$$\hat{q}_E(\tau) = e^{\tau \hat{H}/\hbar}\, \hat{q}\, e^{-\tau \hat{H}/\hbar} \quad \text{such that } \hat{q}_E(0) = \hat{q}(0)$$

DEF The thermal Euclidean n-point correlation function is defined as
$$\langle \hat{q}_E(\tau_n) \cdots \hat{q}_E(\tau_1) \rangle_\beta \equiv Z^{-1}(\beta)\, \mathrm{Tr}\{ e^{-\beta \hat{H}}\, \hat{q}_E(\tau_n) \cdots \hat{q}_E(\tau_1) \} \text{ with } \tau_1 < \cdots < \tau_n$$

↳ The 2-pt function reads
$$\langle \hat{q}_E(\tau_2)\, q_E(\tau_1) \rangle_\beta = Z^{-1}(\beta)\, \mathrm{Tr}\{ e^{-(\hbar\beta - \tau_2)\hat{H}/\hbar}\, \hat{q}\, e^{-(\tau_2-\tau_1)\hat{H}/\hbar}\, \hat{q}\, e^{-\tau_1 \hat{H}/\hbar} \}$$
For non degenerated energy eigenstates, one finds ($\mathrm{Tr}\{\bullet\} \leftrightarrow \langle n|\bullet|n\rangle$)
$$\langle q_E(\tau_2)\, q_E(\tau_1) \rangle_\beta = Z^{-1}(\beta) \sum_{n,m} e^{-(\hbar\beta - \tau_2)E_n/\hbar} \langle n|\hat{q}|m\rangle e^{-(\tau_2-\tau_1)E_m/\hbar} \langle m|\hat{q}|n\rangle e^{-\tau_1 E_n/\hbar}$$
$$= Z^{-1}(\beta) \sum_{n,m} e^{-(\hbar\beta - \tau_2 + \tau_1)E_n/\hbar} |\langle n|\hat{q}|m\rangle|^2 e^{-(\tau_2-\tau_1)E_m/\hbar}$$

→ Taking the low temperature limit $\beta \to \infty$, we get $Z(\beta) \xrightarrow{\beta\to\infty} e^{-\beta E_0}$ and
$$\langle q_E(\tau_2)\, q_E(\tau_1) \rangle_\beta \xrightarrow{\beta\to\infty} \sum_{m\geq 0} |\langle 0|\hat{q}|m\rangle|^2 \exp\{ -(\tau_2-\tau_1)\frac{(E_m - E_0)}{\hbar} \}$$

PROP
$$\lim_{\beta\to 0} \langle q_E(\tau_2)\, q_E(\tau_1) \rangle_\beta = \langle 0| \hat{q}_E(\tau_2)\, \hat{q}_E(\tau_1) |0\rangle$$
The low temperature limit of the thermal n-points correlation function gives us the scattering theory; namely where particle interactions are studied in a vacuum, independant of thermal effect.

↳ For the 1-pt function: $\lim_{\beta\to\infty} \langle \hat{q}_E(\tau) \rangle_\beta = \langle 0|\hat{q}|0\rangle$

DEF The connected 2-pt function is defined as
$$\langle \hat{q}_E(\tau_2)\, \hat{q}_E(\tau_1) \rangle_\beta^c \equiv \langle \hat{q}_E(\tau_2)\, \hat{q}_E(\tau_1) \rangle_\beta - \langle \hat{q}_E(\tau_2) \rangle_\beta \langle \hat{q}_E(\tau_1) \rangle_\beta$$

→ In the low temp. limit, the connected 2-pt fct reads
$$\sum_{m\geq 0} \mapsto \sum_{m>0}$$

$\hookrightarrow \lim_{\beta\to\infty} \langle q_E(\tau_2)\, q_E(\tau_1)\rangle_\beta^c = \sum_{m>0} |\langle 0|\hat{q}|m\rangle| e^{-(\tau_2-\tau_1)(E_m-E_0)/\hbar}$

Thus, for large Euclidean time differences $\tau_2 \gg \tau_1$, we find

$$\lim_{\substack{\beta\to\infty\\ \tau_2-\tau_1\to\infty}} \langle q_E(\tau_2)\, q_E(\tau_1)\rangle_\beta^c \simeq |\langle 0|\hat{q}|1\rangle| e^{-(\tau_2-\tau_1)(E_1-E_0)/\hbar}$$

PROP | The quantities E_1-E_0 (energy gap) and $|\langle 0|\hat{q}|1\rangle|^2$ (transition probability) can be read off from the asymptotics of the connected 2-pt function.

2.6.3 Massive scalar field:

→ Consider $S = \int d^4x \left(\frac{-1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2\right)$

→ In section (2.6.1), for a single HO, we found

$$Z(\beta) = e^{-\alpha\hbar\omega\beta/2}/(1-e^{-\hbar\beta\omega})$$

Since a massive scalar field is a superposition of decoupled HO with frequencies $\omega(\vec{k}) = \sqrt{\vec{k}^2+m^2}$, the partition function of a massive scalar field in d dimension is given by

$$Z(\beta) = \prod_{\vec{k}} \frac{e^{-\alpha\omega(\vec{k})\beta}}{1-e^{-\beta\omega(\vec{k})}} \quad (\text{setting } \hbar = 1)$$

DEF | The free energy $F(\beta)$ is defined through

$$Z(\beta) = e^{-\beta F(\beta)} \iff F(\beta) = -\beta^{-1}\ln Z(\beta)$$

→ One gets $F(\beta) = \sum_{\vec{k}} \left\{ \underbrace{\frac{\alpha\omega(\vec{k})}{2}}_{\text{vacuum energy}} + \beta^{-1}\ln\left(1-e^{-\beta\omega(\vec{k})}\right)\right\}$

In the limit of a large box, $\sum_{\vec{k}} \mapsto \int d^dk/(2\pi)^d$, so that

$$F(\beta)/V = \int d^dk/(2\pi)^d \cdot \left(\underbrace{\alpha\omega(\vec{k})/2}_{\text{diverging}} + \underbrace{\beta^{-1}\ln\left(1-e^{-\beta\omega(\vec{k})}\right)}_{\text{converging}}\right)$$

DEF | In low $T=\beta^{-1}$ expansion, one separates zero point energy $(F/V)_0$ from thermal term $(F/V)_T$:

$$\left(\frac{F(\beta)}{V}\right)_0 \equiv \int \frac{d^dk}{(2\pi)^d}\frac{\alpha\omega(\vec{k})}{2} \quad \text{and} \quad \left(\frac{F(\beta)}{V}\right)_T \equiv \int \frac{d^dk}{(2\pi)^d}\beta^{-1}\ln\left(1-e^{-\beta\omega(\vec{k})}\right)$$